NAVSWC TR 91-321

INTERVAL GRAPHS AND HYPERGRAPH ACYCLICITY DEGREE

BY A.D. PARKS
STRATEGIC SYSTEMS DEPARTMENT

JUNE 1991

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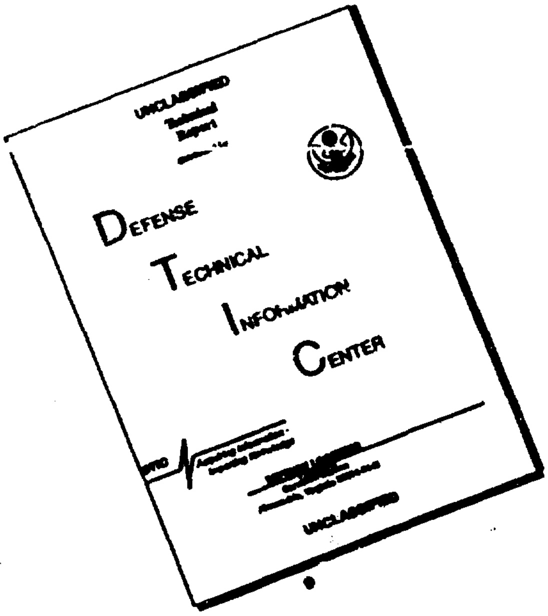

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FOREWORD

This report describes a certain "tangential" theoretical result obtained from work performed in the Strategic Systems (K) Department at the Naval Surface Warfare Center (NAVSWC) as part of an independent research grant entitled "A Category Theoretic Approach to Relational Database Schemes."

This report has been reviewed and approved by Ted Sims, Space Sciences Branch Head, and James L. Sloop, Space and Surface Systems Division Head.

Approved by:

R. L. Schmidt

R. L. SCHMIDT, Head
Strategic Systems Department

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INTRODUCTION

In recent years much research has been devoted to the study of θ-*acyclic* ($\theta = \alpha, \beta, \gamma$, Berge) hypergraphs and their utility as relational database scheme models. It has been shown that such schemes enjoy certain desirable properties (e. g., monotone join expressions); that there is a linear ordering of the strengths of θ-acyclicity (i.e., Berge-acyclicity $\Rightarrow$ γ-acyclicity $\Rightarrow$ β-acyclicity $\Rightarrow$ α-acyclicity; none of the reverse implications hold); and that strengthening the "degree" of acyclicity also strengthens the related desirable properties [**1,2,4,7**]. Furthermore, it is known that a reduced hypergraph is α-acyclic if, and only if, it is conformal and its associated graph is chordal [**1,4**]. The purpose of this report is to exhibit that a reduced conformal hypergraph is β-acyclic if its associated graph is an interval graph.

DEFINITIONS

An *interval graph* is a graph with vertex set V and edge set E for which there exists a mapping ϕ from V into a family of intervals on a line such that for all $u \neq v \in V$, $(u, v) \in E$ if, and only if, $\phi(u) \cap \phi(v) \neq \emptyset$. A graph is *chordal* if every cycle of length ≥ 4 has a chord. It should be noted that every interval graph is chordal [**6**].

A *clique* in a graph is a subgraph of the graph which is complete. A clique is *maximal* if it is not contained in a larger clique. Let $C_1, C_2, \ldots, C_m$ be a ranking of the maximal cliques C_i, $1 \leq i \leq m$, of a graph. Such a ranking is said to be consecutive if whenever a vertex v is in C_j and C_k for $j < k$, then v is in C_ℓ for all ℓ where $j < \ell < k$. The relative ranking of C_j and C_k is denoted by $C_j \prec C_k$ if $j < k$.

A *hypergraph* $\mathcal{H}$ is a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite set of vertices and $\mathcal{E}$ is a set of (hyper)edges which are nonempty subsets of $\mathcal{N}$. The hypergraph $\mathcal{H}$ is reduced if no edge is a subset of another edge. The *associated graph* $G(\mathcal{H})$ of $\mathcal{H}$ is the graph with vertex set $V = \mathcal{N}$ with two vertices in $G(\mathcal{H})$ adjacent if $\mathcal{H}$ has an edge containing both vertices. (It is interesting to note that G is a covariant functor from the category of hypergraphs to the category of graphs [**3**]). The hypergraph $\mathcal{H}$ is *conformal* if for every clique in $G(\mathcal{H})$ there is an edge of $\mathcal{H}$ which contains its vertices.

Let $(S_1, S_2, \ldots, S_m, S_{m+1})$ be a sequence of edges in $\mathcal{H}$, with $S_1, S_2, \ldots, S_m$ distinct, $m \geq 3$, and $S_{m+1} = S_1$. This sequence is a *Graham cycle* if each $S_i \cap S_{i+1} \neq \emptyset$ $(1 \leq i \leq m)$ and $(S_i \cap S_{i+1}) \not\subseteq (S_j \cap S_{j+1})$ and $(S_j \cap S_{j+1}) \not\subseteq (S_i \cap S_{i+1}) (i \neq j)$. Finally, $\mathcal{H}$ is *β-acyclic* if it has no Graham cycle [4].

PRELIMINARY LEMMAS

Several lemmas are required to construct the proof of the main theorem. The following two are due to Beeri, et al. [1] and Fulkerson, et al. [5], respectively.

LEMMA 1. *The hypergraph $\mathcal{H}$ is reduced and conformal if, and only if, its hyperedges are precisely the maximal cliques of $G(\mathcal{H})$.*

LEMMA 2. *A graph is an interval graph if, and only if, there is a ranking of its maximal cliques which is consecutive.*

The next two lemmas follow directly from the definition of a consecutive ranking of maximal cliques:

LEMMA 3. *The list $C_1, C_2, \ldots, C_m$ is a consecutive ranking of maximal cliques if, and only if, for each i, $2 \leq i \leq m$, $(C_i \cap C_1) \subseteq (C_i \cap C_2) \subseteq \cdots \subseteq (C_i \cap C_{i-1})$.*

PROOF: $(\Rightarrow)$ Let C_j, C_k, and C_i be maximal cliques in a consecutive ranking $C_1, C_2, \ldots,$ C_m of maximal cliques such that $1 \leq j \leq k < i \leq m$. If vertex $v \in C_j \cap C_i$, then $v \in C_k$. Thus, $v \in (C_i \cap C_j)$, $(C_i \cap C_k)$ and $(C_i \cap C_j) \subseteq (C_i \cap C_k)$ for $2 \leq i \leq m$. $(\Leftarrow)$ Let $\{C_1, C_2, \ldots, C_m\}$ be a set of maximal cliques such that for $2 \leq i \leq m$, $(C_i \cap C_1) \subseteq (C_i \cap C_2) \subseteq \cdots \subseteq (C_i \cap C_{i-1})$. Then for any i,j,k such that $1 \leq j \leq k < i \leq m$, $(C_i \cap C_j) \subseteq (C_i \cap C_k)$. If vertex $v \in (C_i \cap C_j)$, then $v \in (C_i \cap C_k)$. Thus $v \in C_j, C_k, C_i$ and C_j, C_k, C_i is a consecutive ranking.∎

LEMMA 4. *If $S \subseteq R$, where $|S| \geq 2$ and R is a set of maximal cliques with consecutive ranking, then S has a consecutive ranking.*

PROOF: Let $C_1, C_2, \ldots, C_m$ be the consecutive ranking of R and let $S = \{C_{j_1}, C_{j_2}, \ldots,$ $C_{j_n}\}$ where $1 \leq j_1 < j_2 < \cdots < j_n \leq m$ and $n \geq 2$. Since R has a consecutive ranking, then for each j_i, $2 \leq i \leq n$,

$$(C_{j_i} \cap C_1) \subseteq (C_{j_i} \cap C_2) \subseteq \cdots \subseteq (C_{j_i} \cap C_{j_i - 1}).$$

Thus, for each j_i,

$$(C_{j_i} \cap C_{j_1}) \subseteq (C_{j_i} \cap C_{j_2}) \subseteq \cdots \subseteq (C_{j_i} \cap C_{j_{i-1}}).$$

It follows from Lemma 3 that $C_{j_1}, C_{j_2}, \ldots, C_{j_n}$ is a consecutive ranking of S.∎

MAIN RESULT

THEOREM 5. *Let $\mathcal{H}$ be a reduced conformal hypergraph. If $G(\mathcal{H})$ is an interval graph, then $\mathcal{H}$ is β-acyclic.*

PROOF: Let $G(\mathcal{H})$ be an interval graph ($\mathcal{H}$ is then obviously α-acyclic since $G(\mathcal{H})$ is chordal) and assume that $\mathcal{H}$ is not β-acyclic. There is then a sequence $(S_1, S_2, \ldots, S_m, S_{m+1})$ of distinct hyperedges in $\mathcal{H}$, with $m \geq 3$ and $S_{m+1} = S_1$, which is a Graham cycle. Thus, from Lemma 1 there is an associated set of maximal cliques $\{C_1, C_2, \ldots, C_m\}$ in $G(\mathcal{H})$ with $C_{m+1} = C_1$ such that for all $i \neq j$, $(C_i \cap C_{i+1}) \nsubseteq (C_j \cap C_{j+1})$ and $(C_j \cap C_{j+1}) \nsubseteq (C_i \cap C_{i+1})$. Since $G(\mathcal{H})$ is an interval graph, from Lemma's 2, 3, and 4 there is a consecutive ranking $C_{j_1}, C_{j_2}, \ldots, C_{j_m}$ of these maximal cliques such that

$$(C_{j_m} \cap C_{j_1}) \subseteq (C_{j_m} \cap C_{j_2}) \subseteq \cdots \subseteq (C_{j_m} \cap C_{j_{m-1}}).$$

But this contradicts the Graham cycle non-containment requirements for these cliques, since, if C_0 is defined to be C_m, it follows that $(C_{j_m} \cap C_{j_m - 1}) \subseteq (C_{j_m} \cap C_{j_m + 1})$ if $C_{j_m - 1} \prec C_{j_m + 1}$ or $(C_{j_m} \cap C_{j_m + 1}) \subseteq (C_{j_m} \cap C_{j_m - 1})$ if $C_{j_m + 1} \prec C_{j_m - 1}$. Therefore, $\mathcal{H}$ has no Graham cycles and must be β-acyclic.∎

REMARKS

As suggested by the previous theorem, if a collection of data has attributes with a natural interval graph representation, then it is possible to organize these data into a β-acyclic relational scheme (e.g., tracking data collected over time intervals $T_i = [t_i, t_i']$, $i = 1, 2, \ldots, n$, with attributes T_i and T_j, $i \neq j$, in the same relation if $T_i \cap T_j \neq \emptyset$). This may be an expedient design if resource allocation and deconfliction are important (conflicting attributes are in the same relation); frequent join operations are required (β-acyclic schemas and their subschema enjoy monotone join expressions); and/or the databases are distributed (β-acyclic databases have full reducers).

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It is shown that if the associated graph of a hypergraph is an interval graph, then the hypergraph is beta-acyclic. Advantages related to database scheme design are highlighted.

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